Second-Order Integral Sliding Mode Control for Uncertain Systems With Control Input Time Delay Based on Singular Perturbation Approach

Xiaoyu Zhang, Hongye Su, and Renquan Lu

***Abstract*—This note proposes a robust integral sliding mode (RISM) manifold and a corresponding design method for stabilization control for uncertain systems with control input time delay. Delay-independent sufficient condition for the existence of the RISM surface is given in terms of LMI. An improved sliding mode control (SMC), which is delay-dependent and suitable for small input time delay, keeps system stay on the neighborhood of the RISM surface in finite time. With uncertainties and disturbances, the reaching condition for the neighborhood of the RISM is satisfied. Furthermore, the control makes second-order sliding mode. Based on a numerical example, the proposed method is verified to be efficient and feasible.**

***Index Terms*—Control input time delay, integral sliding mode, LMI, robust control.**

I. Introduction

Time delays of control systems are found abundantly in practical industry processes. Their aftereffect property poses great challenges in designing stable controller, and those difficulties are often the cause of poor performance [1]. Many control approaches for time delay systems (TDS) have been developed. Within these approaches, robust control for TDS has attracted increasing interests since 1990s, especially the using of linear matrix inequality (LMI) techniques with Lyapunov-Krasovskii functional method [2] for TDS with matching or mismatching uncertainties [3].

Sliding mode control (SMC) [4] has been applied to almost all kinds of TDS due to its inherent advantages such as easy implementation, fast response and especially the insensitivity to uncertainties and disturbances [5]. Many results such as [6]–[11] are presented. Generally, SMC utilizes a discontinuous (switching) control law to force trajectory of system state onto a pre-designed sliding manifold. Hence, the desired performance such as stability and robustness can be guaranteed. The discontinuous (switching) control law is built in the sign of the sliding manifold value that may change sign if there is time delay in its control input channel. Therefore, the SMC control design often fails to guarantee the reachability of the sliding manifold due to the control input delay.

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Recently, the input delay problem of SMC design for TDS has been studied. For example, R. Xia designed an integral sliding mode manifold for the uncertain systems with input time delay and uncertainties [12]. X. Li discussed the stability of dynamical systems with input-delay, using Poincar Map to construct the second-order sliding mode control algorithm and analyzed the stability conditions [13]. Yuanqing Xia dealt with a SMC design for LTI systems with both input and state time-varying delays [14] and provided delay-dependent sufficient conditions for the existence of sliding manifold in terms of linear matrix inequalities. J. Chen studied the combination of loop transfer recovery (LTR) observer and sliding mode control (SMC) to solve input time delay by utilizing non-singular linear transformation [15]. In [16], a design approach of SMC for TDS with only the input time delay item $u(t-\tau)$ was proposed, in which singular perturbation approach was used and ultimate boundedness solutions of the closed-loop system were achieved with disturbances. Study results partially resolve the SMC control problem for TDS with input delay, where LMI techniques are widely adopted for its simple structure and convenient applicability.

There are, however, some limitations in the above-mentioned studies. First, most of them focused on systems with both contol signal $u(t)$ and time delay item $u(t-\tau)$, where τ is the delay time. Yet it is not easy to design SMC in the space of state variables for TDS with only the time delay item $u(t-\tau)$. Second, as the hereafter features make the control input signal or state trajectory deterioration, chattering in the closed-loop system becomes more serious.

In this technical note, SMC design for TDS with only the time delay control item is addressed, especially when there exist uncertainties of system matrices and disturbance. A second-order robust integral sliding mode (RISM) control is designed for uncertain systems with control input time delay, and the final control is continuous. The contribution of the note is primary twofold:

1) The reaching phase is eliminated by a robust integral sliding mode (RISM) designed for a class of uncertain systems with input time delay. A RISM design result is presented.
2) A second-order SMC design is achieved by improving the controller design results of [16]. Therefore, the continuous SMC control signal is obtained, namely chattering-free is achieved.

The reminder of this note is as follows. Section II presents the formulation of the considered TDS and some preliminaries. Main results are given in Section III, including an RISM and its corresponding SMC control design. A simple example about practical concentration control problem is given in Section IV, as well as simulation results validating the design in this note. Conclusions are summarized in Section V.

II. Formulation and Preliminary

Consider a class of uncertain system with time-varying input time-delay

$$\begin{aligned}\dot{x}(t) &= [A_0+\Delta A_0(t)]\,x(t)+[B+\Delta B(t)]\,[u\,(t-\tau(t))+w(t)]\\ u(t) &= 0, t \in [-\max\,(\tau(t))\,,0]\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbf{R}^n$ is the system state vector, $u \in \mathbf{R}^m$ is the control input, $A_0 \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are matrices with suitable dimensions, which may have uncertainties such as $\Delta A_0(t)$, $\Delta B(t)$ with suitable dimensions, $w(t) \in \mathbf{R}^m$ is the matched disturbance, $\tau(t) \in \mathbf{R}$ represents the time delay in the control input signal of the system.

It is assumed that the uncertainties do not meet matching conditions but satisfy the following constructed assumptions.

Assumption 1: The uncertainties $\Delta A_0(t)$ and $\Delta B(t)$ can be described by

$$\Delta A_0(t) = H_{a0}F_{a0}(t)E_{a0},\ \Delta B(t) = H_{b0}F_{b0}(t)E_{b0} \tag{2}$$

where $H_{a0} \in \mathbf{R}^{n\times r_a}$, $H_{b0} \in \mathbf{R}^{n\times r_b}$, $E_{a0} \in \mathbf{R}^{r_a\times n}$, and $E_{b0} \in \mathbf{R}^{r_b\times m}$ are known constant matrices, $F_{a0}(t) \in \mathbf{R}^{r_a\times r_a}$ and $F_{b0}(t) \in \mathbf{R}^{r_b\times r_b}$ are Lebesgue-measurable with $r_a, r_b \in \mathbf{R}$ and satisfy $F_{a0}^T(t)F_{a0}(t) \le I, F_{b0}^T(t)F_{b0}(t) \le I$.

Assumption 2: The time derivative of the uncertainties $\Delta A_0(t)$ and $\Delta B(t)$ are all bounded, that satisfy

$$\|\Delta A_0'(t)\| \le \psi_1,\ \|\Delta B'(t)\| \le \psi_2$$

where ψ_1, ψ_2 are positive scalars.

Assumption 3: The disturbance $w(t)$ and its time derivative are all bounded, that satisfy

$$\|w(t)\| \le \epsilon,\ \|\dot{w}(t)\| \le \bar{\epsilon}$$

where ϵ, $\bar{\epsilon}$ are positive scalars.

Suppose that the system matrices and time delay satisfy the following assumptions.

Assumption 4: $[A_0, B]$ is controllable and $\operatorname{rank}(B) = m$, i.e., B is with full rank.

Assumption 5: The uncertain input time delay $\tau(t)$ satisfies

$$0 < \tau(t) \le h < \infty,\ \dot{\tau}(t) \le d < 1 \tag{3}$$

where h and d are positive constant scalars.

The following preliminaries will be used to derive results.

Lemma 1: For arbitrary matrix $A \in \mathbf{R}^{n\times n}$, if $\|A\| < 1$ then $I - A$ is not singular and satisfies that $\|(I - A)^{-1}\| \le \|I\|/1 - \|A\|$ with I an identity matrix [17].

Lemma 2: For arbitrary matrices A and B with suitable dimensions, if A and $I + BA^{-1}$ are not singular, then $(A + B)^{-1} = A^{-1} - A^{-1}(I + BA^{-1})^{-1}BA^{-1}$ [17].

Lemma 3: Let H and E be real constant matrices with appropriate dimensions. $F(t)$ satisfies $F^T(t)F(t) \le I$, then for any scalar $\varepsilon > 0$, $HF(t)E + E^TF^T(t)H^T \le \varepsilon HH^T + \varepsilon^{-1}E^TE$ holds [18].

III. Main Results

A. Sliding Manifold Design

Define an ISM as the following:

$$Z(t) = Cx(t) - Cx(0) - \int_0^t [(CA_0 - K)x(t) + QZ(t)]\, dt \tag{4}$$

where $K \in \mathbf{R}^{m\times n}$, $Q \in \mathbf{R}^{m\times m}$ are the parameter matrices to be designed, $C \in \mathbf{R}^{m\times n}$ is a known coefficient that satisfies the following assumption.

Assumption 6: There always exists a parameter matrix $C \in \mathbf{R}^{m\times n}$ which satisfies that $0 \le \|C\Delta B(t)\| \le \gamma\|CB\|$, where the scalar γ satisfies the inequality $0 \le \gamma < 1$.

Remark 1: The integral sliding mode (4) satisfies $Z(0) = 0$ which implies that the system state runs on the sliding surface from the initial time instant.

The sliding mode $Z(t)$ is expected to be converging to zero, i.e., $Z(t) = 0$. Furthermore, the system behavior is restricted to the sliding mode, which can be described by equation (1) and $\dot{Z}(t) = 0$ as follows:

$$\begin{aligned} \dot{x}(t) &= (A_0 + \Delta A_0(t))\, x(t) + (B + \Delta B(t))\, u_q\,(t - \tau(t)) \\ 0 &= C\Delta A_0(t)x(t) + Kx(t) \\ &\quad + C\,(B + \Delta B(t))\,(u_q\,(t - \tau(t)) + w(t)) \end{aligned} \tag{5}$$

where $u_q(t - \tau(t))$ is the equivalent control input of the corresponding SMC when $Z(t) = 0$. Combined with (5), it can be further derived that

$$\begin{aligned} u_q\,(t - \tau(t)) = &-(CB + C\Delta B(t))^{-1} \\ &\times \{C\Delta A_0(t)x(t) + Kx(t)\} - w(t). \end{aligned} \tag{6}$$

Remark 2: The controller (6) cannot be implemented because: a) there is time delay in control signal equation, resulting unavailable state signal unless predictor is constructed; and b) there are uncertainties and disturbances in (6).

Then by Lemma 2 and the equation (6), equation (5) becomes

$$\dot{x}(t) = \left[\hat{A}_0 + (\hat{C} - NC)\Delta A_0(t) - NK\right] x(t) \tag{7}$$

where

$$\begin{aligned} N &= (\Delta B(t) - BM - \Delta B(t)M)\,(CB)^{-1} \\ M &= (CB)^{-1}\left[I + C\Delta B(t)(CB)^{-1}\right]^{-1} C\Delta B(t) \\ \hat{A}_0 &= A_0 - B(CB)^{-1}K \\ \hat{C} &= I - B(CB)^{-1}C. \end{aligned} \tag{8}$$

The first result of designing a RISM surface is given as follows.

Theorem 1: The state of the system (1) restricted to the ISM $Z(t) = 0$ (the (5)) is robust quadratically stable if there exist matrix $K \in \mathbf{R}^{m\times n}$, symmetric positive definite matrix P satisfy the following LMI:

$$\begin{bmatrix} L & E_{a0}^T & \bar{E}_{b0}^T \\ E_{a0} & -\frac{\varepsilon}{4}I_{n\times n} & 0 \\ \bar{E}_{b0} & 0 & -\frac{\varepsilon}{3}I_{r_b\times r_b} \end{bmatrix} < 0 \tag{9}$$

for some scalar $\varepsilon > 0$, where

$$L = \hat{A}_0^TP + P\hat{A}_0 + \varepsilon P\Omega P,\ \bar{E}_{b0} = E_{b0}(CB)^{-1}K \tag{10}$$

and the matrix Ω is determined by

$$\begin{aligned} \Omega = \hat{C}H_{a0}H_{a0}^T\hat{C}^T &+ \frac{\delta_1 + \delta_2}{(1-\gamma)^2}B(CB)^{-1}(CB)^{-T}B^T \\ &+ \left[1 + \delta_1 + \frac{(\delta_1 + \delta_2)\delta_3}{(1-\gamma)^2}\right] H_{b0}H_{b0}^T \end{aligned} \tag{11}$$

with the scalars $\delta_i (i = 1, 2, 3)$ are maximum eigenvalues of the following matrices:

$$\begin{aligned} \delta_1 &= \lambda_{\max}\left[E_{b0}(CB)^{-1}CH_{b0}H_{b0}^TC^T(CB)^{-T}E_{b0}^T\right] \\ \delta_2 &= \lambda_{\max}\left[CH_{b0}H_{b0}^TC^T\right] \\ \delta_3 &= \lambda_{\max}\left[E_{b0}(CB)^{-1}(CB)^{-T}E_{b0}^T\right]. \end{aligned} \tag{12}$$

The proof of Theorem 1 is provided in Appendix A.

Remark 3: In the RISM (4), the parameter Q is the self-feedback item of $Z(t)$. It should be positive definite in order to guarantee the stability. Meanwhile, it does not affect the RISM. However, it is critical for controller design, which will be considered in later phases of controller design.

B. An Improved Controller Design

From the sliding manifold (4), it is easily to obtain that

$$\dot{Z}(t) = C\Delta A_0(t)x(t) + Kx(t) - QZ(t) + C\left(B + \Delta B(t)\right)\left(u\left(t - \tau(t)\right) + w(t)\right) \quad (13)$$

which denotes that the control $u(t-\tau(t))$ can depicted as

$$u\left(t - \tau(t)\right) = \left(CB + C\Delta B(t)\right)^{-1}\left[\dot{Z}(t) + QZ(t) - C\Delta A_0(t)x(t) - Kx(t)\right] - w(t). \quad (14)$$

Substitute the control $u(t-\tau(t))$ into (1), the following can be obtained:

$$\dot{x}(t) = \hat{A}_0 x(t) + B(CB)^{-1}\left(\dot{Z}(t) + QZ(t)\right) + \phi_1(t) \quad (15)$$

where

$$\phi_1(t) = \left[(\hat{C} - NC)\Delta A_0(t) - NK\right]x(t) + N\left[\dot{Z}(t) + QZ(t)\right]. \quad (16)$$

Then continuously differentiate of (13) with respect to time t

$$\ddot{Z}(t) = K\hat{A}_0 x(t) + KB(CB)^{-1}\left(\dot{Z}(t) + QZ(t)\right) - Q\dot{Z}(t) + CB\dot{u}\left(t - \tau(t)\right) + \phi_2(t) \quad (17)$$

is derived in (17), where

$$\phi_2(t) = \left[C\Delta A_0(t)x(t)\right]' + \left[C\left(B + \Delta B(t)\right)w(t)\right]' + \left[C\Delta B(t)u\left(t - \tau(t)\right)\right]' + K\phi_1(t). \quad (18)$$

Consequently by defining

$$z_1(t) = x(t),\ z_2(t) = \left[Z(t), \dot{Z}(t)\right]^T$$

we can get

$$\begin{aligned}\dot{z}_1(t) &= \left(A_{11} + \Delta A_{11}(t)\right)z_1(t) + \left(A_{12} + \Delta A_{12}(t)\right)z_2(t)\\ \dot{z}_2(t) &= A_{21}z_1(t) + A_{22}z_2(t) + B_2\left[CB\dot{u}\left(t - \tau(t)\right) + \phi_2(t)\right]\end{aligned} \quad (19)$$

where the parameter matrices

$$\begin{aligned} A_{11} &= \hat{A}_0,\ A_{12} = \left[\, B(CB)^{-1}Q \quad B(CB)^{-1} \,\right]\\ A_{21} &= \begin{bmatrix} 0 \\ K\hat{A}_0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}\\ A_{22} &= \begin{bmatrix} 0 & I_m \\ KB(CB)^{-1}Q & KB(CB)^{-1} - Q \end{bmatrix}\end{aligned} \quad (20)$$

and the uncertain matrices

$$\Delta A_{11}(t) = (\hat{C} - NC)\Delta A_0(t) - NK,\ \Delta A_{12}(t) = N\bar{Q} \quad (12)$$

with $\bar{Q} = [Q\ I_m]$.

A controller of the form

$$\dot{u}(t) = -(CB)^{-1}\left[\frac{Z(t)}{\mu} + (1 + \delta)\bar{\phi}\operatorname{sign}Z(t)\right] \quad (22)$$

can be designed for system (19), where $\mu > 0$ and $\delta > 0$ and both are tuning parameters. The parameter $\bar{\phi}$ is used to compensate the uncertainty $\phi_2(t)$, which is given by (41) in Appendix C.

The closed-loop system given by (19) and (22) has the following form:

$$\begin{aligned}\dot{z}_1(t) &= \left(A_{11} + \Delta A_{11}(t)\right)z_1(t) + \left(A_{12} + \Delta A_{12}(t)\right)z_2(t)\\ \mu\dot{z}_2(t) &= \mu A_{21}z_1(t) + \mu A_{22}z_2(t) - \hat{B}z_2\left(t - \mu\xi(t)\right) + \mu B_2\bar{w}(t)\end{aligned} \quad (23)$$

where the constant matrix $\hat{B} = B_2[I_m \quad 0], \mu\xi(t) = \tau(t), 0 \leq \xi(t) \leq h$, and

$$\bar{w}(t) = \phi_2(t) - (1 + \delta)\bar{\phi}\operatorname{sign}Z\left(t - \mu\xi(t)\right) \quad (24)$$

which implies $\bar{w}(t)$ is boundary as described in Appendix C.

For all small positive scalar μ, a system defined by (23) is a singularly perturbed system. The time delay is scaled by μ in order to guarantee robust stability with respect to small enough delay. This depiction is also used by Han[16] and Fridman [19].

However, the dynamic equation of $z(t) = [z_1(t), z_2(t)]^T$ in (19) is different from [16]. Especially there are more uncertainties in $\Delta A_{11}(t)$, $\Delta A_{12}(t)$ and more complicated disturbance $\phi_2(t)$. Moreover, the control signal is $\dot{u}(t - \tau(t))$.

Improved design of parameters μ, δ of continuous sliding mode control (22) guarantees that the system state is driven onto the given RISM.

Theorem 2: Given positive tuning scalars μ, h, α and b, if there exist matrices $Q \in \mathbb{R}^{m\times m} > 0$, $P_1 \in \mathbb{R}^{n\times n} > 0$, $P_2 \in \mathbb{R}^{n\times 2m}$ and positive definite matrices $P_3, G, R, S \in \mathbb{R}^{2m\times 2m}$, such that the following LMI:

$$\Theta_\mu = \begin{bmatrix} \theta_{11} & \cdots & \theta_{16} \\ \vdots & \ddots & \vdots \\ * & \cdots & \theta_{66} \end{bmatrix} < 0 \quad (25)$$

where

$$\begin{aligned}\theta_{11} =& P_1A_{11} + A_{11}^TP_1 + \mu P_2^TA_{21} + \mu A_{21}P_2 + \alpha P_1\\ &+ \varepsilon' P_1\Omega P_1 + 4\varepsilon'^{-1}E_{a0}^TE_{a0} + \frac{\eta_1}{\alpha}I_n\\ \theta_{12} =& P_1A_{12} + \mu A_{21}^TP_3 + \mu A_{11}^TP_2^T + \mu P_2^TA_{22} + \alpha\mu P_2^T\\ &+ 3\varepsilon'^{-1}K^T(CB)^{-T}E_{b0}^TE_{b0}(CB)^{-1}\bar{Q}\\ \theta_{14} =& -P_2^T\hat{B}, \theta_{15} = P_2^T\hat{B}, \theta_{16} = h\mu A_{21}^TR\\ \theta_{22} =& \mu(P_2A_{12} + A_{12}^TP_2^T + P_3A_{22}\\ &+ A_{22}^TP_3 + \alpha P_3) + G + S - e^{-\alpha\mu h}R\\ &+ 3\varepsilon'^{-1}\bar{Q}^T(CB)^{-T}E_{b0}^TE_{b0}(CB)^{-1}\bar{Q} + \frac{\eta_2}{\alpha}I_{2m}\\ \theta_{24} =& -P_3 + e^{-\alpha\mu h}R,\ \theta_{25} = P_3\ \theta_{26} = \mu hA_{22}^TR\\ \theta_{33} =& -e^{-\alpha\mu h}G - e^{-\alpha\mu h}R,\ \theta_{34} = e^{-\alpha\mu h}R\\ \theta_{44} =& -2e^{-\alpha\mu h}R - (1 - d)e^{-\alpha\mu h}S,\ \theta_{46} = -h\hat{B}^TR\\ \theta_{55} =& -bI_{2m},\ \theta_{56} = h\hat{B}^TR,\ \theta_{66} = -R\end{aligned}$$

is feasible for some positive scalar ε'. Then solutions of the closed-loop system (23) satisfy

$$z^T(t)P_\mu z(t) < e^{-\alpha(t-t_0)}z^T(t_0)P_\mu z(t_0) + \frac{\mu^2b}{\alpha}\|\bar{w}_{[t_0,t]}\|_\infty^2 \quad (26)$$

where

$$P_\mu = \begin{bmatrix} P_1 & \mu P_2^T \\ & \mu P_3 \end{bmatrix} > 0,\ P_\mu \in \mathbb{R}^{(n+2m)*(n+2m)} \quad (27)$$

for all $\xi(t) \in [0,\ h]$ with $\mu\dot{\xi}(t) \leq d < 1$. Moreover, solutions of the closed-loop system (23) satisfy (26) for all fast-varying delays $\xi(t) \in [0,\ h]$ if LMI (25) is feasible with $S = 0$.

The proof of Theorem 2 is provided in Appendix B.

The controller design conditions of the overall closed-loop system, one refers to X.Han's contribution[16]. The following remarks are proposed in view of the proposed controller described by (22) for the closed-loop system defined in (19).

Remark 4: if $\Delta A_{11}(t) = 0$ and $\Delta A_{12}(t) = 0$ namely $\Delta A_0(t)$ and $\Delta B_0(t)$ vanish, Theorem 2 is the same as [16]. However, the sliding manifold is not in linear but integral sliding mode, which makes the closed-loop equation (19) different. Due to this, the designed controller (22) holds the capacities of treating uncertainties.

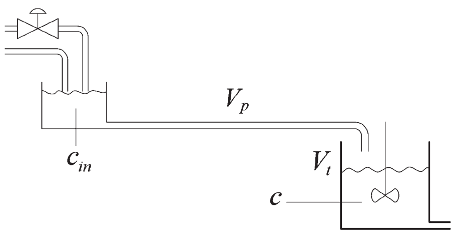

Fig. 1. Schematic diagram of the concentration control system.

Remark 5: The controller (22) is of the time derivative form. It indicates that the SMC controller is designed on the derivative of the control signal $u(t)$. The final controller is in the following form:

$$u(t)=u(t_0)-\int_{t_0}^{t}(CB)^{-1}\left[\frac{Z(t)}{\mu}+(1+\delta)\bar{\phi}\text{sign}Z(t)\right]dt$$

where the initial value of $u(t)$ could be simply set $u(t_0)=0$. As a result, the proposed controller is a continuous sliding mode control.

Remark 6: Under the proposed controller (22) the sliding manifold $Z(t)$ and $\dot{Z}(t)$ are in the small neighborhood of zero. The bound of the neighborhood is directly related with the controller parameter $\bar{\phi}$. The ideal sliding mode $Z(t)=0$ and $\dot{Z}(t)=0$ can be achieved, only if the uncertainties and disturbance vanish and the controller parameter $\bar{\phi}$ is selected as zero. This shows that the sliding mode of the closed-loop system is a second-order sliding mode.

IV. Numerical Example

Consider a concentration control problem for a fluid, which flows through a pipe to a tank in which the fluid was perfect mixed. A schematic diagram of the process is shown in Fig. 1. The mass balance description is given by

$$V_t\frac{dc(t)}{dt}=[q+\Delta q(t)]\left[c_{in}\left(t-\tau(t)\right)-c(t)+w(t)\right]+n(t) \quad (28)$$

where $c_{in}(t)$ is the concentration at the inlet of the pipe, $c(t)$ is the concentration at the outlet of the tank, q is the flow rate and $\Delta q(t)$ its corresponding uncertainty, $w(t)$ is the concentration disturbance such as leakage and non-homogenization. $n(t)$ stands for white noise which power is 10^{-4}. Let V_t be the tank volume, V_p be the pipe volume, and then the flow time delay $\tau(t)$ is given by

$$\tau(t)=\frac{V_p}{q+\Delta q(t)}. \quad (29)$$

In the mass balance, $c_{in}(t)$ is the control input. $V_t=10$, $V_p=1, q=1, \Delta q(t)=0.6\sin 8\pi t, w(t)=0.54\sin 8\pi t+0.1e^{-0.48t}$. The system parameters can be considered as $A_0=-0.1$, $B=0.1$, $H_{a0}=-0.9$, $H_{b0}=0.6$, $E_{a0}=0.67$, $E_{b0}=1$, $F_{a0}(t)=F_{b0}(t)=\sin 8\pi t$, and $\tau(t)\approx 1-0.6\sin 8\pi t$.

We select $C=1$ to satisfy Assumption 4 and design the parameter of the RISM K as 0.24 based on Theorem 1. We then select appropriate values for control input parameters according to Theorem 2. The control input is

$$c_{in}(t)=-\int_{0}^{t}[1760Z(t)+0.9\text{sign}Z(t)]dt.$$

The sliding mode is

$$Z(t)=c(t)-c(0)-\int_0^t[720Z(t)-0.34c(t)]dt.$$

And we design a SMC controller via linear sliding mode surface by using the method of [16], in order to validate our method. Its formulation is $c_{in}(t)=-3.2c(t)-0.9\ \text{sign}c(t)$. The simulation results of step response under the two controllers are shown in Figs. 1–4, in case that $c(0)=0$.

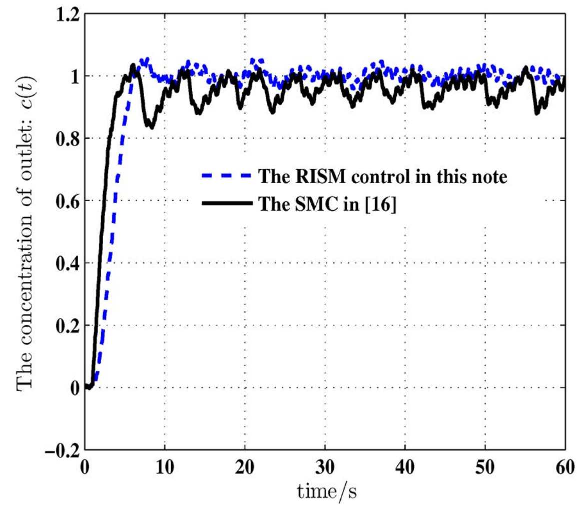

Fig. 2. Outlet concentration curves $c(t)$.

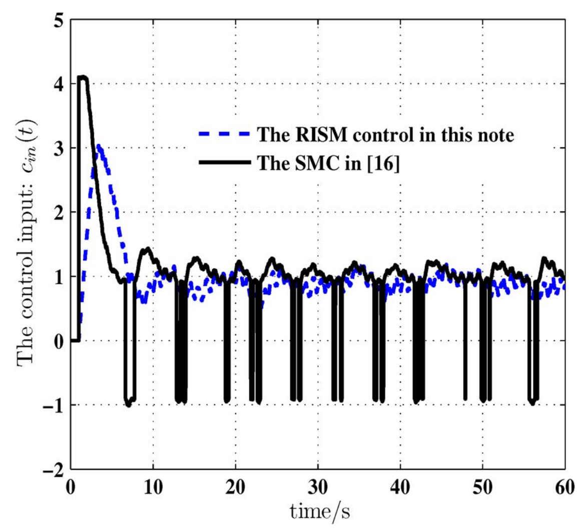

Fig. 3. Control input curves $cin(t)$.

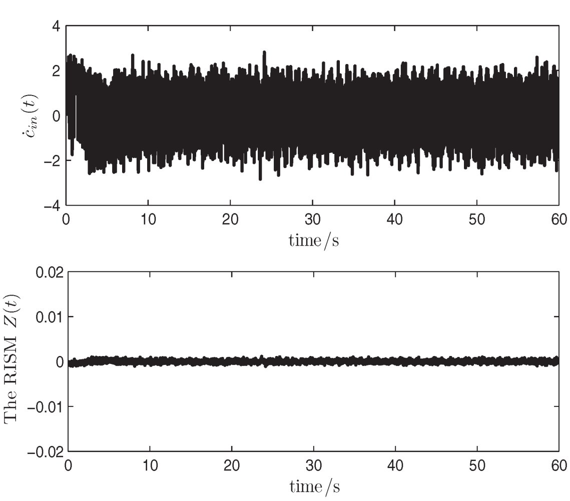

Fig. 4. Derivative signal of control input $\dot{c}_{cin}(t)$ and the RISM $Z(t)$.

From Fig. 3, we found that the time delay shifted the control input signal and in Fig. 2 it can be seen that the state curve under the SMC controller of [16] had fluctuations. Fortunately, the state curve under our controller floated slightly. The integral sliding manifold value $Z(t)$ in Fig. 4 diverged during the delay time. After the delay time, the control input proposed by this note drove the concentration $c(t)$ onto the sliding mode. It can preserve the reachability of the integral sliding mode although there are uncertainties, and therefore the sliding manifold value was small.

Fig. 4 shows that the time derivative signal of our controller was switching. However, it is shown in Fig. 3 that the control input signal of our controller was continuous, while the control input signal of the SMC controller of [16] had chattering obviously.

The control design in this note guarantees the reachability of $Z(t) = 0$ and $\dot{Z}(t) = 0$. Therefore, the sliding mode is second-order sliding mode.

V. Conclusion

Integral sliding mode (ISM) surface and the corresponding stabilization control design are presented for uncertain systems with control input time-delay, using singular perturbation approach (SPA). Sufficient condition for the existence of an ISM surface, as well as the SMC control design which can drive the system state onto the ISM, are all given in terms of LMIs. The sliding mode control is designed on the derivative of control input signal. At the same time, it is guaranteed that the ISM and its first order time derivative remain in the vicinity of origin. Therefore, second-order sliding mode is achieved, and the uncertainties are not required to satisfy the matching conditions. The final control is continuous. Simulation results of a numerical example show that the proposed method is effective and feasible.

Appendix I

Choose a Lyapunov function as $V(x(t),t) = x^T(t)Px(t)$, and calculate its derivative along the trajectory of system state equation (7) with respect to the time t. The following can be obtained:

$$\dot{V}(x(t),t) = x^T(t)Wx(t)$$

where

$$W = \hat{A}_0^T P + P\hat{A}_0 + \Delta L_1 + \Delta L_2 + \Delta L_3 \tag{30}$$

$$\begin{aligned}\Delta L_1 &= P\hat{C}H_{a0}F_{a0}(t)E_{a0} + E_{a0}^T F_{a0}^T(t) H_{a0}^T \hat{C}^T P^T\\ \Delta L_2 &= \Delta L_{2,1} + \Delta L_{2,2} + \Delta L_{2,3}\\ \Delta L_3 &= \Delta L_{3,1} + \Delta L_{3,2} + \Delta L_{3,3}\end{aligned} \tag{31}$$

with its components by

$$\begin{aligned}\Delta L_{2,1} =& P\Delta B(t)(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}+\\ &\left(P\Delta B(t)(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}\right)^T\\ \Delta L_{2,2} =& PBM(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}+\\ &\left(PBM(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}\right)^T\\ \Delta L_{2,3} =& P\Delta B(t)M(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}+\\ &\left(P\Delta B(t)M(CB)^{-1}CH_{a0}F_{a0}(t)E_{a0}\right)^T\\ \Delta L_{3,1} =& P\Delta B(t)(CB)^{-1}K + \left(P\Delta B(t)(CB)^{-1}K\right)^T\\ \Delta L_{3,2} =& PBM(CB)^{-1}K + \left(PBM(CB)^{-1}K\right)^T\\ \Delta L_{3,3} =& P\Delta B(t)M(CB)^{-1}K + \left(P\Delta B(t)M(CB)^{-1}K\right)^T.\end{aligned} \tag{32}$$

Define a matrix $\Lambda = [I + C\Delta B(t)(CB)^{-1}]^{-1}$. According to Assumption 4 and Lemma 1, both Λ^{-1} and Λ are all not singular and satisfy that

$$\|\Lambda\| \le \frac{1}{1-\gamma}. \tag{33}$$

By Lemma 3 and equations (8), (31), (32), (33) the following inequalities hold for any scalar $\varepsilon_1 > 0$ or $\varepsilon_{i,j} > 0 (i = 2,3,\ j = 1,2,3)$

$$\begin{aligned}\Delta L_1 &\le \varepsilon_1 P\bar{H}_{a0}\bar{H}_{a0}^T P^T + \varepsilon_1^{-1} E_{a0}^T E_{a0}\\ \Delta L_{2,1} &\le \delta_1\varepsilon_{2,1} PH_{b0}H_{b0}^T P^T + \varepsilon_{2,1}^{-1} E_{a0}^T E_{a0}\\ \Delta L_{2,2} &\le \frac{\delta_1\varepsilon_{2,2}}{(1-\gamma)^2} PH_u H_u^T P^T + \varepsilon_{2,2}^{-1} E_{a0}^T E_{a0}\\ \Delta L_{2,3} &\le \frac{\delta_1\delta_3\varepsilon_{2,3}}{(1-\gamma)^2} PH_{b0}H_{b0}^T P^T + \varepsilon_{2,3}^{-1} E_{a0}^T E_{a0}\\ \Delta L_{3,1} &\le \varepsilon_{3,1} PH_{b0}H_{b0}^T P^T + \varepsilon_{3,1}^{-1} \bar{E}_{b0}^T \bar{E}_{b0}\\ \Delta L_{3,2} &\le \frac{\delta_2\varepsilon_{3,2}}{(1-\gamma)^2} PH_u H_u^T P^T + \varepsilon_{3,2}^{-1} \bar{E}_{b0}^T \bar{E}_{b0}\\ \Delta L_{3,3} &\le \frac{\delta_2\delta_3\varepsilon_{3,3}}{(1-\gamma)^2} PH_{b0}H_{b0}^T P^T + \varepsilon_{3,3}^{-1} \bar{E}_{b0}^T \bar{E}_{b0}\end{aligned}$$

where $\bar{H}_{a0} = \hat{C}H_{a0}$, $H_u = B(CB)^{-1}$. Accordingly

$$\Delta L \le P\Omega_0 P + \alpha_1^{-1} E_{a0}^T E_{a0} + \alpha_2^{-1} \bar{E}_{b0}^T \bar{E}_{b0} \tag{34}$$

where $\Omega_0 = \alpha_h \bar{H}_{a_0}\bar{H}_{a_0}^T + \alpha_b \bar{H}_{b_0}\bar{H}_{b_0}^T + \alpha_u \bar{H}_u \bar{H}_u^T$ with the scalars

$$\begin{aligned}\alpha_h &= \varepsilon_1,\ \alpha_b = \varepsilon_{3,1} + \delta_1\varepsilon_{2,1} + \frac{\delta_1\delta_3\varepsilon_{2,3}}{(1-\gamma)^2} + \frac{\delta_2\delta_3\varepsilon_{3,3}}{(1-\gamma)^2}\\ \alpha_u &= \frac{\delta_1\varepsilon_{2,2}}{(1-\gamma)^2} + \frac{\delta_2\varepsilon_{3,2}}{(1-\gamma)^2}\\ \alpha_1^{-1} &= \varepsilon_1^{-1} + \varepsilon_{2,1}^{-1} + \varepsilon_{2,2}^{-1} + \varepsilon_{2,3}^{-1},\ \alpha_2^{-1} = \varepsilon_{3,1}^{-1} + \varepsilon_{3,2}^{-1} + \varepsilon_{3,3}^{-1}.\end{aligned}$$

We can choose the scalars $\varepsilon_0 = \varepsilon_1 = \varepsilon$ and $\varepsilon_{i,j} (i = 1,2,\ j = 1,2,3) = \varepsilon$, then $\Omega_0 = \varepsilon\Omega$. Hereby, the matrixinequality $W < 0$ is equivalent to $L_0 + \Delta L < 0$. Hence, by equation (34), $W < 0$ is equivalent with the inequality (9). The proof is completed.

Appendix II

Choose a Lyapunov-Krasovskii function of the following form:

$$V_\mu(t) = z^T(t)P_\mu z(t) + \mu h \int_{-\mu h}^{0}\int_{t+\lambda}^{t} e^{\alpha(s-t)} \dot{z}_2^T(s)R\dot{z}_2(s)\,ds\,d\lambda + \int_{t-\mu h}^{t} e^{\alpha(s-t)} z_2^T(s)Gz_2(s)\,ds + \int_{t-\mu\xi(t)}^{t} e^{\alpha(s-t)} z_2^T(s)Sz_2(s)\,ds$$

where G, R and $S \in \mathbb{R}^{2m}$ are positive definite matrices. Based on to this function, the following is obtained:

$$\begin{aligned}W'(t) &= \frac{d}{dt}V_\mu(t) + \alpha V_\mu(t) - \mu^2 b\bar{w}^T(t)\bar{w}(t)\\ &= W(t) + z^T(t)\Xi(t)z(t),\end{aligned} \tag{35}$$

where $W(t)$ concludes the expanded items of the equation (35) refers to Appendix A of [16] and

$$\Xi(t) = \Delta\bar{L}_1 + \Delta\bar{L}_2 + \Delta\bar{L}_3$$

with the matrices $\Delta\bar{L}_1,\ \Delta\bar{L}_2$, and $\Delta\bar{L}_3$ are described as

$$\begin{aligned}\Delta\bar{L}_1 &= P_\mu \begin{bmatrix} \hat{C}\Delta A_0 & (t)0\\ 0 & 0\end{bmatrix} + \begin{bmatrix} \hat{C}\Delta A_0 & (t)0\\ 0 & 0\end{bmatrix}^T P_\mu\\ \Delta\bar{L}_2 &= P_\mu \begin{bmatrix} NC\Delta A_0(t) & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix} NC\Delta A_0 & (t)0\\ 0 & 0\end{bmatrix}^T P_\mu\\ \Delta\bar{L}_3 &= P_\mu \begin{bmatrix} NK & N\bar{Q}\\ 0 & 0\end{bmatrix} + \begin{bmatrix} NK & N\bar{Q}\\ 0 & 0\end{bmatrix}^T P_\mu.\end{aligned}$$

Denoting

$$\tilde{H}_a = \begin{bmatrix} H_{a0} & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{H}_b = \begin{bmatrix} H_{b0} & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{E}_a = \begin{bmatrix} E_{a0} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{E}_b = \begin{bmatrix} E_{b0}(CB)^{-1} & KE_{b0}(CB)^{-1}\bar{Q} \\ 0 & 0 \end{bmatrix},\ \tilde{H}_u = \begin{bmatrix} B(CB)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

by the proof of Theorem 1, we can get

$$\Xi(t) \le \varepsilon' P_\mu \bar{\Omega} P_\mu^T + 4\varepsilon'^{-1}\tilde{E}_{a0}^T\tilde{E}_{a0} + 3\varepsilon'^{-1}\tilde{E}_{b0}^T\tilde{E}_{b0}$$

for some positive scalar ε', where

$$\bar{\Omega} = \tilde{H}_a\tilde{H}_a^T + \frac{\delta_1+\delta_2}{(1-\gamma)^2}\tilde{H}_u\tilde{H}_u^T + \left[1+\delta_1+\frac{(\delta_1+\delta_2)\delta_3}{(1-\gamma)^2}\right]\tilde{H}_b\tilde{H}_b^T$$

with the scalars $\delta_i (i = 1, 2, 3)$ are maximum eigenvalues denoted by the equation (12).

Then, base on Appendix A in [16], by setting

$$\zeta(t) = col\left[z_1(t), z_2(t), z_2(t-\mu h), z_2\left(t-\mu\xi(t)\right), \mu\bar{w}(t)\right]$$

and applying Schur complements to the term $\mu^2\ h^2\dot{z}_2^T(t)R\dot{z}_2(t)$ on the right-hand side of the equation (35), $W'(t) < 0$ holds if $\Theta_\mu < 0$. The proof is completed.

Appendix III

According to (18), by using (14), (15), (16) and the controller (22), we have

$$\begin{aligned}\phi_2(t) =& C\left[\Delta A_0'(t)+\Delta A_0(t)\hat{A}_0-\Delta B'(t)(CB+C\Delta B(t))^{-1}\right.\\ &\left.\times\left(C\Delta A_0(t)+K\right)\right]x(t)\\ &+C\left[\Delta A_0(t)B(CB)^{-1}+\Delta B'(t)(CB+C\Delta B(t))^{-1}\right]\\ &\times\left[\dot{Z}(t)+QZ(t)\right]\\ &+\left[C\Delta A_0(t)+K\right]\left[(\hat{C}-NC)\Delta A_0(t)-NK\right]x(t)\\ &+\left[C\Delta A_0(t)+K\right]N\left[\dot{Z}(t)+QZ(t)\right]\\ &+C\left(B+\Delta B(t)\right)\dot{w}(t)-\mu^{-1}C\Delta B(t)(CB)^{-1}Z(t)\\ &-C\Delta B(t)(CB)^{-1}(1+\delta)\bar{\phi}\mathrm{sign}Z(t).\end{aligned}$$

Consequently, by Assumption 2 and Assumption 3 we can get the bound of $\phi_2(t)$, namely

$$\|\phi_2(t)\| \le \eta_0 + \eta_1\|z_1(t)\| + \eta_2\|z_2(t)\| \tag{36}$$

where

$$\eta_0 = \bar{\epsilon}\left\|C\left(B+\Delta B(t)\right)\right\| + (1+\delta)\bar{\phi}\left\|C\Delta B(t)(CB)^{-1}\right\| \tag{37}$$

$$\begin{aligned}\eta_1 =& \|C\|\left[\psi_1+\left\|\Delta A_0(t)\hat{A}_0\right\|\right.\\ &\left.+\psi_2\left\|(CB+C\Delta B(t))^{-1}\left(C\Delta A_0(t)+K\right)\right\|\right]\\ &+\left\|\left[C\Delta A_0(t)+K\right]\left[(\hat{C}-NC)\Delta A_0(t)-NK\right]\right\|\end{aligned} \tag{38}$$

$$\begin{aligned}\eta_2 =& (1+\|Q\|)\left\{\left\|C\Delta A_0(t)B(CB)^{-1}\right\|\right.\\ &\left.+\psi_2\left\|(CB+C\Delta B(t))^{-1}\right\|+\left\|\left[C\Delta A_0(t)+K\right]N\right\|\right\}.\end{aligned} \tag{39}$$

Remark 7: Clearly, the boundary parameters η_0, η_1, η_2 are limited by the boundaries of uncertainties $\Delta A_0(t)$, $\Delta B(t)$ and their time derivatives $\Delta A_0'(t)$ and $\Delta B'(t)$.

We then aim to decide the parameter $\bar{\phi}$ of the controller (22). In the closed-loop system (23), the parameter $\bar{\phi}$ is used to compensate the uncertainty $\phi_2(t)$. Therefore, it should be chosen as $(1+\delta)\bar{\phi} \ge \eta_0$. By (37), we have

$$(1+\delta)\bar{\phi} \ge \bar{\epsilon}\left[1-\left\|C\Delta B(t)(CB)^{-1}\right\|\right]^{-1}\left\|C\left(B+\Delta B(t)\right)\right\|. \tag{40}$$

According to Assumption 4, $\|C\Delta B(t)(CB)^{-1}\| \le \gamma$. Consider this relationship and Assumption 1; the parameter $\bar{\phi}$ can be given as

$$\bar{\phi} \ge \frac{\bar{\epsilon}}{1-\gamma}\left[\|CB\| + \|CH_{b0}\|\|E_{b0}\|\right]. \tag{41}$$

Finally, the boundary of $\bar{w}(t)$ is given as

$$\begin{aligned}\|\bar{w}(t)\| &\le \|\phi_2(t)\| + (1+\delta)\sqrt{m}\bar{\phi}\\ &\le \eta_1\|z_1(t)\| + \eta_2\|z_2(t)\| + (1+\delta)(1+\sqrt{m})\bar{\phi}.\end{aligned} \tag{42}$$

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